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Coulomb corrections in quasi-elastic scattering based on the eikonal expansion for electron wave functions

J. A. Tjon[1,2] and S. J. Wallace[2]

[1]*Physics Department, University of Utrecht, 3508 TA Utrecht, The Netherlands*
[2]*Department of Physics, University of Maryland, College Park, MD 20742*

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An eikonal expansion is developed in order to provide systematic corrections to the eikonal approximation through order $1/k^2$, where k is the wave number. The expansion is applied to wave functions for the Klein-Gordon equation and for the Dirac equation with a Coulomb potential. Convergence is rapid at energies above about 250 MeV. Analytical results for the eikonal wave functions are obtained for a simple analytical form of the Coulomb potential of a nucleus. They are used to investigate distorted-wave matrix elements for quasi-elastic electron scattering from a nucleus. Focusing factors are shown to arise from the corrections to the eikonal approximation. A precise form of the effective-momentum approximation is developed by use of a momentum shift that depends on the electron's energy loss.

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I. INTRODUCTION

Quasielastic scattering of electrons by nuclei can provide important information about the response of a nucleus to a weakly interacting probe. Experiments have been performed at the MIT Bates Laboratory [1–9], at the Saclay Laboratory [10–15] and at SLAC [16–18] in order to explore this reaction. A recent review of quasielastic scattering provides a guide to the experimental and theoretical results [19]. One important theoretical issue concerns the corrections that arise from the Coulomb potential of the nucleus. Another issue is the accuracy with which longitudinal (L) and transverse (T) response functions can be determined when Coulomb corrections are present.

In order to include the Coulomb distortion of electron waves, it is necessary to solve the Dirac equation for scattering of an electron of mass m and energy E from a nucleus, i.e.,

$$\Big(E - \alpha \cdot \mathbf{p} - V(r) - \beta m\Big)\psi = 0, \tag{1}$$

where $V(r)$ is the Coulomb potential due to the nuclear charge distribution. Exact solutions for the Dirac-Coulomb wave functions may be obtained as a sum over partial waves. [20–22] However, the sum involves delicate numerical computations. As the electron energy increases, the partial-wave expansions converge more slowly in spite of the fact that the Coulomb corrections become smaller. A simpler approach is to use the eikonal approximation for the distortions because the approximation gets better as the energy increases and it can provide insight into the parameters that control the Coulomb corrections.

For waves moving along the z-direction, the eikonal approximation provides an approximate solution,

$$\psi \approx e^{ikz} e^{i\chi}, \tag{2}$$

where $k = \sqrt{E^2 - m^2}$ is the momentum and χ is a simple function of the potential. It is important to include focusing factors such as those derived by Yennie, Boos and Ravenhall [23–25] based upon the WKB approximation to the distorted Coulomb waves. Some particulary transparent results have been obtained using the eikonal approximation to derive an effective-momentum approximation (ema) [26, 27] that produces results very similar to plane-wave results. However, with the combination of the eikonal approximation, WKB focusing factors and the effective-momentum approximation, the approach lacks a systematic basis and accuracy is uncertain. Approximations to the partial-wave analysis, such as the analysis of Ref [28], that attempt to improve the effective-momentum approximation also have uncertain accuracy. Significant disagreements in the determination of nuclear response functions from experimental data [19, 29] have arisen at least in part owing to the use of different theoretical methods to remove the Coulomb corrections.

In order to address the issue of Coulomb corrections, we develop corrections to the eikonal approximation based upon a systematic expansion about the high-energy limit. This eikonal expansion is shown to be rapidly convergent for typical energies and targets used in quasi-elastic scattering. For a simple analytical Coulomb potential, analytical forms are developed for the eikonal corrections through second-order in $1/k$, where k is the electron's wave number.

The eikonal approximation has a long history [30]. Glauber used the eikonal approximation to develop a simple and insightful form of multiple scattering theory that is called Glauber theory [31]. Czyz and Gottfreid [32] used the eikonal approximation to analyze electron scattering but that work did not include focusing factors. Work by Giusti *et al.* also is based on the eikonal approximation [33, 34] and some recents works have combined the eikonal approximation with semi-classical focusing factors in order to assess Coulomb corrections in quasi-elastic scattering. [35, 36]. We show that the focusing factors arise from systematic corrections to the eikonal approximation.

Corrections to the eikonal approximation also have a long history. Work by Saxon and Schiff [37] showed how to correct the approximation to leading order in $1/k$. A systematic expansion for the scattering t-matrix was developed by Sugar and Blankenbecler [38]. Systematic corrections to the Glauber approximation were developed by Wallace [39] and extended to the Dirac scattering amplitude in [40]. However, a systematic expansion for wave functions has not been developed prior to this work.

In Section II, we develop the eikonal expansion for Klein-Gordon wave functions because that is the simplest and most transparent case. In Section III, we develop the eikonal expansion for the Dirac wave function. Section III A focuses on $u(\mathbf{r})$, which is a Pauli spinor containing the two upper components of the Dirac wave function. Because there is a spin-orbit interaction, additional spin-dependent terms arise in the eikonal expansion. Because the analysis is technically more complicated, details are given in an appendix. In Section III B we show that the Pauli spinor $\ell(\mathbf{r})$ that contains the two lower components of the Dirac wave function takes a very simple form in the limit of vanishing electron mass, i.e., $\ell(\mathbf{r}) = 2\lambda u(\mathbf{r})$, where $\lambda = \pm\frac{1}{2}$ is the helicity. This follows because of the structure of the Dirac equation in the limit of vanishing electron mass. Conservation of the electron's helicity holds to high accuracy at electron energies of interest. Section III C discusses convergence of the eikonal expansion and analytical results for the eikonal phases, which are given in an appendix. Section III D discusses the different focusing factors that arise for Klein-Gordon and Dirac waves. In Section IV, we consider helicity matrix elements of the electron current based on the Dirac distorted waves. We show that the spin-orbit distortions arising from the Coulomb interaction can alter the longitudinal and transverse helicity matrix elements in somewhat different ways, which can affect the L/T separation. Specializing to the longitudinal response, section V discusses quasi-elastic scattering by use of a simple model of the nuclear response. Section VI revisits the effective-momentum approximation (*ema*) and Section VII presents results of calculations of the longitudinal response function. The analytical eikonal wave functions with systematic corrections included are used to describe the Coulomb corrections. We show that the effective-momentum approximation can be made precise by use of a calculated momentum shift that depends on the electron's energy loss, ω. A summary and some conclusions are presented in Section VIII.

II. EIKONAL EXPANSION FOR KLEIN-GORDON WAVES

For the Klein-Gordon equation with a Coulomb potential $V(r)$, one has

$$\Big([E - V(r)]^2 - \mathbf{p}^2 - m^2\Big)\psi(\mathbf{r}) = 0. \tag{3}$$

The potential may be a point-like Coulomb potential or a potential that is derived from a finite charge distribution, such as that of a nucleus. Writing the wave function in the form of a plane wave propagating in the z-direction with wave number $k = \sqrt{E^2 - m^2}$ and a complex phase shift $\bar{\chi}$,

$$\psi(\mathbf{r}) = e^{ikz} e^{i\bar{\chi}(\mathbf{r})}, \tag{4}$$

leads to the following equation for the phase shift,

$$[E - V(r)]^2 - [k\hat{z} + \nabla\bar{\chi} - i\nabla]\cdot[k\hat{z} + \nabla\bar{\chi}] - m^2 = 0. \tag{5}$$

Using the fact the $E^2 - k^2 - m^2 = 0$ and dividing by $2k$ leads to a differential equation for $\bar{\chi}$,

$$\frac{\partial\bar{\chi}}{\partial z} = -\frac{V}{v} + \frac{V^2}{2k} - \frac{(\nabla\bar{\chi})^2}{2k} + i\frac{\nabla^2\bar{\chi}}{2k}, \tag{6}$$

where $v = k/E \approx 1$ when $E >> m$.

For outgoing-wave boundary conditions and a potential that decreases faster than $1/r$, the wave function must be a pure plane wave as $z \to -\infty$ and a phase-shifted plane wave as $z \to \infty$. For a Coulomb potential, the same boundary condition is used with the understanding that the potential is cut off at a large distance $r > \Lambda$. For either case, the outgoing-wave eikonal phase vanishes as $z \to -\infty$ and the eikonal phase is found by integrating Eq. (6) as follows,

$$\begin{aligned}\bar{\chi}^{(+)}(\mathbf{r}) \;=\; & -\frac{1}{v}\int_{-\infty}^{z} dz' V(r') \\ & -\frac{1}{2k}\int_{-\infty}^{z} dz' \Big((\nabla'\bar{\chi}^{(+)}(\mathbf{r}'))^2 - V^2\Big) \\ & +\frac{i}{2k}\int_{-\infty}^{z} dz' \nabla'^2\bar{\chi}^{(+)}(\mathbf{r}').\end{aligned} \tag{7}$$

A superscript (+) denotes the outgoing-wave boundary condition.

The eikonal expansion is an iterative solution of Eq. (7) about the limit $k \to \infty$. It initially takes the form

$$\bar{\chi}^{(+)} = \bar{\chi}_0^{(+)} + \bar{\chi}_1^{(+)} + \bar{\chi}_2^{(+)} + \cdots, \tag{8}$$

where the "barred" phase shifts are complex and their subscripts denote the order of iteration. The lowest order term is appropriate to the limit $k \to \infty$ and is obtained from the first term on the right side of Eq. (7),

$$\bar{\chi}_0^{(+)}(\mathbf{r}) = -\frac{1}{v}\int_{-\infty}^{z} dz' V(r'), \tag{9}$$

where $r' = \sqrt{z'^2 + b^2}$ and b is the impact parameter.

Higher order terms in the expansion are evaluated as follows. In first order, one evaluates the left side of Eq. (7) using $\bar{\chi}_0^{(+)} + \bar{\chi}_1^{(+)}$ and the right side using $\bar{\chi}_0^{(+)}$. After cancelling terms that are equal because of Eq. (9), the first-order term $\bar{\chi}_1^{(+)}$ is obtained as

$$\bar{\chi}_1^{(+)}(\mathbf{r}) = -\frac{1}{2k}\int_{-\infty}^{z} dz' \left((\nabla' \bar{\chi}_0^{(+)}(\mathbf{r}'))^2 - V^2 \right) + \frac{i}{2k}\int_{-\infty}^{z} dz' \nabla'^2 \bar{\chi}_0^{(+)}(\mathbf{r}'). \quad (10)$$

In second order, one evaluates the left hand side of Eq. (7) using $\bar{\chi}_0^{(+)} + \bar{\chi}_1^{(+)} + \bar{\chi}_2^{(+)}$ and the right hand side using $\bar{\chi}_0^{(+)} + \bar{\chi}_1^{(+)}$. After cancelling terms that are equal because of Eqs. (9) and (10), the second-order term is obtained as

$$\bar{\chi}_2^{(+)}(\mathbf{r}) = -\frac{1}{2k}\int_{-\infty}^{z} dz' \Big[2\nabla' \bar{\chi}_0^{(+)}(\mathbf{r}') \cdot \nabla' \bar{\chi}_1^{(+)}(\mathbf{r}') + (\nabla' \bar{\chi}_1^{(+)}(\mathbf{r}'))^2 \Big] + \frac{i}{2k}\int_{-\infty}^{z} dz' \nabla'^2 \bar{\chi}_1^{(+)}(\mathbf{r}'). \quad (11)$$

Higher-order terms are not considered in this work.

At the final stage of analysis, it is convenient to separate the real and imaginary parts of the eikonal phase as follows,

$$\bar{\chi}^{(+)} = \chi^{(+)} + i\omega^{(+)} \quad (12)$$

and to expand each of these "unbarred" phases in systematic powers of $1/k$,

$$\begin{aligned} \chi^{(+)} &= \chi_0^{(+)} + \chi_1^{(+)} + \chi_2^{(+)} + \cdots \\ \omega^{(+)} &= \omega_1^{(+)} + \omega_2^{(+)} + \cdots . \end{aligned} \quad (13)$$

Here the subscripts denote the power of $1/k$ that is involved. Through second order, the systematically ordered phases for the Klein-Gordon waves are,

$$\begin{aligned} \chi_0^{(+)}(\mathbf{r}) &= -\frac{1}{v}\int_{-\infty}^{z} dz' V(r') \\ \chi_1^{(+)}(\mathbf{r}) &= -\frac{1}{2k}\int_{-\infty}^{z} dz' \Big((\nabla' \chi_0^{(+)}(\mathbf{r}'))^2 - V^2(r') \Big) \\ \chi_2^{(+)}(\mathbf{r}) &= -\frac{1}{2k}\int_{-\infty}^{z} dz' \Big[2\nabla' \chi_0^{(+)}(\mathbf{r}') \cdot \nabla' \chi_1^{(+)}(\mathbf{r}') \\ &+ \quad \nabla'^2 \omega_1^{(+)}(\mathbf{r}') \Big] \\ \omega_1^{(+)}(\mathbf{r}) &= \frac{1}{2k}\int_{-\infty}^{z} dz' \nabla'^2 \chi_0^{(+)}(\mathbf{r}') \\ \omega_2^{(+)}(\mathbf{r}) &= \frac{1}{2k}\int_{-\infty}^{z} dz' \Big[\nabla'^2 \chi_1^{(+)}(\mathbf{r}') \\ &\quad - 2\nabla' \chi_0^{(+)}(\mathbf{r}') \cdot \nabla' \omega_1^{(+)}(\mathbf{r}') \Big] \end{aligned} \quad (14)$$

For the Klein-Gordon wave function with outgoing-wave boundary conditions, this gives

$$\psi^{(+)}(\mathbf{r}) = e^{ikz} e^{i\chi^{(+)}} e^{-\omega^{(+)}}, \quad (15)$$

and one may work at various orders in the eikonal expansion by truncating the expansions of Eq. (14).

Incoming wave boundary conditions are appropriate for final-state wave functions in matrix elements. In that case the wave function must be a pure plane wave as $z \to \infty$ and a phase-shifted plane wave as $z \to -\infty$. It is written as

$$\psi^{(-)}(\mathbf{r}) = e^{ikz} e^{-i\bar{\chi}^{(-)}}, \quad (16)$$

which leads to

$$\frac{\partial \bar{\chi}^{(-)}}{\partial z} = \frac{V}{v} - \frac{V^2}{2k} + \frac{(\nabla \bar{\chi}^{(-)})^2}{2k} + i\frac{\nabla^2 \bar{\chi}^{(-)}}{2k}. \quad (17)$$

Integration with incoming-wave boundary conditions produces

$$\begin{aligned} \bar{\chi}^{(-)}(\mathbf{r}) &= -\frac{1}{v}\int_{z}^{\infty} dz' V(r') \\ &\quad - \frac{1}{2k}\int_{z}^{\infty} dz' \left((\nabla' \bar{\chi}^{(-)}(\mathbf{r}'))^2 - V^2 \right) \\ &\quad - \frac{i}{2k}\int_{z}^{\infty} dz' \nabla'^2 \bar{\chi}^{(-)}(\mathbf{r}'). \end{aligned} \quad (18)$$

As before, the eikonal expansion is an iterative solution of Eq. (18) that produces an expansion about the limit $k \to \infty$. It initially takes the form

$$\bar{\chi}^{(-)} = \bar{\chi}_0^{(-)} + \bar{\chi}_1^{(-)} + \bar{\chi}_2^{(-)} + \cdots . \quad (19)$$

with the lowest order term being

$$\bar{\chi}_0^{(-)}(\mathbf{r}) = -\frac{1}{v}\int_{z}^{\infty} dz' V(r'). \quad (20)$$

Higher order terms in the expansion are evaluated in the same manner as discussed above. They are

$$\begin{aligned} \bar{\chi}_1^{(-)}(\mathbf{r}) &= -\frac{1}{2k}\int_{z}^{\infty} dz' \left((\nabla' \bar{\chi}_0^{(-)}(\mathbf{r}'))^2 - V^2 \right) \\ &\quad - \frac{i}{2k}\int_{z}^{\infty} dz' \nabla'^2 \bar{\chi}_0^{(-)}(\mathbf{r}'), \\ \bar{\chi}_2^{(-)}(\mathbf{r}) &= -\frac{1}{2k}\int_{z}^{\infty} dz' \Big[2\nabla' \bar{\chi}_0^{(-)}(\mathbf{r}') \cdot \nabla' \bar{\chi}_1^{(-)}(\mathbf{r}') \\ &\quad + (\nabla' \bar{\chi}_1^{(-)}(\mathbf{r}'))^2 \Big] - \frac{i}{2k}\int_{z}^{\infty} dz' \nabla'^2 \bar{\chi}_1^{(+)}(\mathbf{r}'). \end{aligned} \quad (21)$$

Separating the real and imaginary parts of the eikonal phases as follows,

$$\bar{\chi}^{(-)} = \chi^{(-)} - i\omega^{(-)}, \quad (22)$$

and expanding each of these "unbarred" phases in systematic powers of $1/k$, we have

$$\begin{aligned}\chi^{(-)} &= \chi_0^{(-)} + \chi_1^{(-)} + \chi_2^{(-)} + \cdots \\ \omega^{(-)} &= \omega_1^{(-)} + \omega_2^{(-)} + \cdots . \end{aligned} \tag{23}$$

Such systematically ordered phases for the Klein-Gordon waves for incoming wave boundary conditions may be obtained from Eq. (14) by replacing all integrations $\int_{-\infty}^{z}$ by $\int_{z}^{\infty}$ and all superscripts (+) by (−). Because of the symmetry of the potential with respect to inversion of z, the following relations hold for each order n,

$$\begin{aligned}\chi_n^{(-)}(z, \mathbf{b}) &= \chi_n^{(+)}(-z, \mathbf{b}), \\ \omega_n^{(-)}(z, \mathbf{b}) &= \omega_n^{(+)}(-z, \mathbf{b}). \end{aligned} \tag{24}$$

For the Klein-Gordon wave function with incoming wave boundary conditions, this gives

$$\psi^{(-)}(\mathbf{r}) = e^{ikz} e^{-i\chi^{(-)}} e^{-\omega^{(-)}}, \tag{25}$$

and one may work at various orders by truncating the expansions for the eikonal phases.

The imaginary part of the eikonal phase produces the "focusing factor",

$$f^{KG(\pm)}(\mathbf{r}) = e^{-\omega^{(\pm)}(\mathbf{r})}. \tag{26}$$

In order to provide some insight into the focusing factor, it is useful to consider a simple potential for which the eikonal phases may be determined analytically. That is done in Appendix A and we find that at $\mathbf{r} = \mathbf{0}$,

$$\omega_1^{(\pm)}(\mathbf{0}) = \frac{V(0)}{2kv} \approx \frac{V(0)}{2E}, \tag{27}$$

where $V(0)$ is the potential at the origin. Thus, the focusing factor is approximately

$$f^{KG(\pm)}(\mathbf{0}) \approx 1 - V(0)/(2E). \tag{28}$$

A matrix element for emission of a photon with momentum $\mathbf{q}$ and energy $\omega = E_i - E_f$ involves initial and final momenta $\mathbf{k}_i$ and $\mathbf{k}_f$. It takes the form

$$\begin{aligned} M^{\ \mu} &= \int d^3 r \psi_{\mathbf{k}_f}^{(-)*}(\mathbf{r}) j_{KG}^{\mu} e^{-i\mathbf{q}\cdot\mathbf{r}} \psi_{\mathbf{k}_i}^{(+)}(\mathbf{r}), \\ &= \int d^3 r e^{i(\mathbf{Q}-\mathbf{q})\cdot\mathbf{r}} e^{i\chi} f_f^{KG(-)}(\mathbf{r}) f_i^{KG(+)}(\mathbf{r}) j_{KG}^{\mu} \end{aligned} \tag{29}$$

where $\mathbf{Q} = \mathbf{k}_i - \mathbf{k}_f$ and $\chi = \chi_f^{(-)} + \chi_i^{(+)}$. It should be noted that $\chi_i^{(+)}$ and $\omega_i^{(+)}$ are obtained from Eqs. (14) with the z-axis parallel to initial momentum $\mathbf{k}_i$, while $\chi_f^{(-)}$ and $\omega_f^{(-)}$ are obtained from the same equations using the outgoing-wave condition ($\int_z^{\infty}$) with the z-direction parallel to final momentum $\mathbf{k}_f$.

The conserved current for the Klein-Gordon equation also contains Coulomb distortions, i.e.,

$$\begin{aligned} j_{KG}^0 &= \left[E_i + E_f - 2V(r)\right]/\sqrt{4E_iE_f}, \\ \mathbf{j}_{KG} &= \left[\mathbf{k}_i + \mathbf{k}_f + \nabla\overline{\chi}_i^{(+)}(\mathbf{r}) - \nabla\overline{\chi}_f^{(-)}(\mathbf{r})\right]/\sqrt{4E_iE_f}, \end{aligned} \tag{30}$$

where phase-space factors for initial and final states have been included. Current conservation holds because

$$\begin{aligned}\left(\omega j_{KG}^0 - \mathbf{q}\cdot\mathbf{j}_{KG}\right)e^{-i\mathbf{q}\cdot\mathbf{r}} \propto e^{-i\mathbf{q}\cdot\mathbf{r}} G^{-1}(E_i) - \\ G^{-1}(E_f)e^{-i\mathbf{q}\cdot\mathbf{r}}, \end{aligned} \tag{31}$$

where $G^{-1}(E) = [E - V(r)]^2 - \mathbf{p}^2 - m^2$ annihilates the wave function at energy E. Relative to the plane-wave current, the Klein-Gordon current j_{KG}^0 contributes to the focusing effect a factor equal to $(1 - V(r)/\bar{E})$ where $\bar{E} = \frac{1}{2}(E_i + E_f)$ is the average energy. The initial-state wave function provides a factor given by Eq. (26), i.e., $f_i^{D(+)} \approx (1 - V(0)/(2\bar{E}))$ and the final state wave function provides a similar factor. The combination of these factors at $\mathbf{r} = 0$ is approximately equal to $(1 - V(0)/\bar{E})^2$. That agrees with the result of Yennie et al. [23] for the Dirac wave function.

In the case of Dirac-Coulomb waves, the conserved electron current is the Dirac matrix γ^{μ}, which it is not modified by the presence of a Coulomb potential. It does not contain any focusing effects. The Dirac focusing factors arise solely from the wave functions and are given in Eq. (51), which yields $f^{D(\pm)} \approx (1 - V(0)/\bar{E})$ for each wave function. Thus, there is a different focusing factor than for a Klein-Gordon wave function. However, the combination of currents and wave functions produces similar overall focusing effects for the Klein-Gordon and Dirac current matrix elements. We note that the analysis of Refs. [35, 36] used the Dirac focusing factors with the Klein-Gordon current, which produces one too many factors of $1 - V(0)/\bar{E}$.

III. EIKONAL EXPANSION FOR DIRAC-COULOMB WAVES

For the Dirac equation, the eikonal expansion is carried out in two stages. First we consider the Pauli spinor $u(\mathbf{r})$ that contains the two upper components of the Dirac wave function, i.e.,

$$\psi(\mathbf{r}) = \begin{pmatrix} u(\mathbf{r}) \\ \ell(\mathbf{r}) \end{pmatrix}. \tag{32}$$

It follows from the Dirac equation that the Pauli spinor $\ell(\mathbf{r})$ that contains the two lower components may be determined in a second stage from

$$\ell(\mathbf{r}) = \frac{1}{E_2 - V}\ \sigma\cdot\mathbf{p}\ u(\mathbf{r}), \tag{33}$$

where $E_2 = E + m$.

A. Upper component spinor

Eliminating the lower component spinor from the Dirac equation leads to the following equation for the upper-component spinor

$$\Big(E_1 - V - \sigma\cdot\mathbf{p}\frac{1}{E_2 - V}\sigma\cdot\mathbf{p}\Big)u(\mathbf{r}), \tag{34}$$

where $E_1 = E - m$. For electron scattering it is generally the case that $E >> m$ and thus $E_1 \approx E_2 \approx E$.

For outgoing-wave boundary conditions, the Pauli spinor $u(\mathbf{r})$ is written in terms of a complex eikonal phase $\bar{\chi}^{(+)}(\mathbf{r})$ and a complex spin-dependent phase $\bar{\gamma}^{(+)}(\mathbf{r})$ as follows

$$u^{(+)}(\mathbf{r}) = \left(1 - \frac{V}{E_2}\right)^{1/2} e^{ikz} e^{i\bar{\chi}^{(+)}} e^{i\sigma_e\bar{\gamma}^{(+)}}. \tag{35}$$

The wave propagates in the z-direction and an impact vector $\mathbf{b}$ is defined as the part of $\mathbf{r}$ that is perpendicular to the $\hat{z}$-direction, i.e., $\mathbf{b} = \hat{z}\times(\mathbf{r}\times\hat{z})$. Three orthogonal unit vectors are : $\hat{z}$, $\hat{b} = \mathbf{b}/|\mathbf{b}|$ and $\hat{e} = \hat{b}\times\hat{z}$. The spin matrix in the eikonal phase is $\sigma_e = \sigma\cdot\hat{e}$. The factor $(1 - V/E_2)^{1/2}$ is introduced in order to sum up terms that otherwise arise in higher orders.

The eikonal expansion is developed in Appendix B. The result is that the eikonal phases are expanded in a systematic fashion in powers of $1/k$ as follows.

$$\begin{aligned}
\chi^{(+)} &= \chi_0^{(+)} + \chi_1^{(+)} + \chi_2^{(+)} + \cdots \\
\omega^{(+)} &= \omega_1^{(+)} + \omega_2^{(+)} + \cdots \\
\gamma^{(+)} &= \gamma_1^{(+)} + \gamma_2^{(+)} + \cdots \\
\delta^{(+)} &= \delta_2^{(+)} + \cdots ,
\end{aligned} \tag{36}$$

where the subscript of each term denotes the power of $1/k$ that is involved. The systematically ordered phases for the Dirac equation are as follows.

$$\begin{aligned}
\chi_0^{(+)}(\mathbf{r}) &= -\frac{1}{v}\int_{-\infty}^{z} dz' V(r'), \\
\chi_1^{(+)}(\mathbf{r}) &= -\frac{1}{2k}\int_{-\infty}^{z} dz'\Big((\nabla'\chi_0^{(+)}(\mathbf{r}'))^2 - V^2(r')\Big), \\
\chi_2^{(+)}(\mathbf{r}) &= -\frac{1}{2k}\int_{-\infty}^{z} dz'\Big[(2\nabla'\chi_0^{(+)}(\mathbf{r}')\cdot\nabla'\chi_1^{(+)}(\mathbf{r}') \\
&+ \nabla'^2\omega_1^{(+)}(\mathbf{r}')\Big] - \frac{1}{4kE_2}\nabla^2\chi_0^{(+)}, \\
\omega_1^{(+)}(\mathbf{r}) &= \frac{1}{2k}\int_{-\infty}^{z} dz'\nabla'^2\chi_0^{(+)}(\mathbf{r}'), \\
\omega_2^{(+)}(\mathbf{r}) &= \frac{1}{2k}\int_{-\infty}^{z} dz'\Big[\nabla'^2\chi_1^{(+)}(\mathbf{r}') \\
&- 2\nabla'\chi_0^{(+)}(\mathbf{r}')\cdot\nabla'\omega_1^{(+)}(\mathbf{r}')\Big], \\
\gamma_1^{(+)}(\mathbf{r}) &= -\frac{1}{2k}\int_{-\infty}^{z} dz'\frac{\partial V(r)}{\partial b} \\
\gamma_2^{(+)}(\mathbf{r}) &= -\frac{1}{2k}\int_{-\infty}^{z} dz'\Bigg(2\nabla'\chi_0^{(+)}(\mathbf{r}')\cdot\nabla'\gamma_1^{(+)}(\mathbf{r}') \\
&- \frac{1}{E}\frac{\partial V}{\partial z}\frac{\partial\chi_0^{(+)}}{\partial b}\Bigg) \\
\delta_2^{(+)}(\mathbf{r}) &= \frac{1}{2k}\int_{-\infty}^{z} dz'\left(\nabla'^2 - \frac{1}{b^2}\right)\gamma_1^{(+)}(\mathbf{r}').
\end{aligned} \tag{37}$$

Results for $\chi_0^{(+)}$, $\chi_1^{(+)}$, $\omega_1^{(+)}$ and $\omega_2^{(+)}$ are the same as for the Klein-Gordon case. The last term in the result for $\chi_2^{(+)}$ is not present in the Klein-Gordon case.

Spin-dependent phases can be shown to be simply related to the spin-independent ones as follows,

$$\gamma_1^{(+)} + \gamma_1^{(+)} + i\delta_2^{(+)} = \frac{1}{E_2 - V}\frac{\partial}{\partial b}\Big[\chi_0^{(+)} + \chi_1^{(+)} + i\omega_1^{(+)}\Big].$$

This connection holds to order $1/k^2$.

The upper-component spinor of the Dirac wave function for helicity λ and outgoing-wave boundary conditions is given by

$$u_\lambda^{(+)} = \Big(1 - \frac{V}{E_2}\Big)^{1/2} e^{ikz} e^{i\chi^{(+)}} e^{-\omega^{(+)}} e^{i\sigma_e\bar{\gamma}_i^{(+)}}\xi_\lambda, \tag{38}$$

where ξ_λ is a helicity eigenstate. One may work at various orders of the eikonal expansion by truncating the expansions of Eq. (36).

The upper-component spinor for helicity λ and incoming-wave boundary conditions is given by

$$u_\lambda^{(-)} = \Big(1 - \frac{V}{E_2}\Big)^{1/2} e^{ikz} e^{-i\chi^{(-)}} e^{-\omega^{(-)}} e^{-i\sigma_e\bar{\gamma}^{(-)}}\xi_\lambda, \tag{39}$$

where

$$\begin{aligned}
\overline{\gamma}^{(-)} &= \gamma^{(-)} - i\delta^{(-)} \\
&= \gamma_1^{(-)} + \gamma_2^{(-)} \cdots - i(\delta_2^{(-)} + \cdots),
\end{aligned} \tag{40}$$

with phases obtained from Eqs. (36) and (37) by replacing the superscripts (+) by (−) and the integration ranges $\int_{-\infty}^{z}$ by $\int_{z}^{\infty}$. Alternatively, the symmetry of Eq. (24) may be used.

B. Lower-component spinors

Equation (33) specifies the lower-component spinor in terms of the upper-component spinor. The connection is very simple in the limit $m \to 0$ when helicity eigenstates are used. We find

$$\ell_\lambda(\mathbf{r}) = 2\lambda u_\lambda(\mathbf{r}) \tag{41}$$

where $\lambda = \pm\frac{1}{2}$. This holds generally in the $m = 0$ limit as the following analysis shows.

For $m = 0$, we have $E_1 = E_2 = E$. If we define $\tilde{u}_\lambda = \Big(1 - V/E\Big)^{1/2} u_\lambda(\mathbf{r})$, then Eq. (34) may be written as

$$\tilde{u}_\lambda(\mathbf{r}) = h^2 \tilde{u}_\lambda, \tag{42}$$

where for $E > |V|$, h is the hermitian operator

$$h \equiv \frac{1}{\sqrt{E-V}}\ \sigma\cdot\mathbf{p}\ \frac{1}{\sqrt{E-V}}. \tag{43}$$

Thus, $\tilde{u}_\lambda$ is an eigenfunction of the hermitian operator h^2 with eigenvalue +1. It must then also be an eigenfunction of h with eigenvalues ±1, that is,

$$h\tilde{u}_\lambda = \pm\tilde{u}_\lambda. \tag{44}$$

However, because of the Dirac equation it follows that

$$h\tilde{u}_\lambda = \tilde{\ell}_\lambda, \tag{45}$$

where $\tilde{\ell}_\lambda = \Big(1-V/E\Big)^{1/2} \ell_\lambda$. These two equations require that

$$\tilde{\ell}_\lambda = 2\lambda\tilde{u}_\lambda \tag{46}$$

where $\lambda = \pm\frac{1}{2}$, which proves Eq. (41). Clearly, h is the helicity operator for the $m = 0$ Dirac equation. Conservation of helicity is a well-known result for the limit $m \to 0$ when the interaction is a vector current, or any single component of a vector interaction as for the Coulomb interaction. In the limit as $V \to 0$, h becomes the usual helicity operator for a plane-wave state.

For a nonzero electron mass, a similar but approximate analysis may be performed to show that

$$\begin{aligned}\ell_\lambda(\mathbf{r}) &\approx 2\lambda\left(\frac{E_1 - \langle V\rangle}{E_2 - \langle V\rangle}\right)^{1/2} u_\lambda(\mathbf{r}),\\ &\approx 2\lambda\Big(1 - \frac{m}{E_2 - \langle V\rangle}\Big)u_\lambda(\mathbf{r}),\end{aligned} \tag{47}$$

where $\langle V\rangle$ is the average potential. The correction term is about one part per thousand for a 500 MeV electron. It will be neglected in this work.

C. Convergence of eikonal expansion

Convergence of the eikonal expansion is discussed for scattering of a 500 MeV electron by ^{208}Pb. Given that the electron mass is $m = .511 MeV$, it follows that $k \approx E$ and $v \approx 1$, both within a part per million. The Coulomb potential is approximately $V(0) = 25$ MeV at the center of the nucleus. The leading order eikonal phase is of order $2RV(0)$, where R is the mean radius and the factor $2R$ provides an estimate of the integral over z. If the charge distribution is approximated as constant within a sphere of radius R, then $V(0) = \frac{3}{2}Z\alpha/R$, so we expect that $\chi_0^{(+)} \approx 3Z\alpha$. For ^{208}Pb, this gives $\chi_0^{(+)} \approx 2$. The eikonal expansion introduces corrections that involve the nondimensional ratio $V/E \approx .05$, so we expect $\chi_2^{(+)} \approx .0025\chi_0^{(+)} \approx .005$. The eikonal expansion is not convergent but is asymptotic, meaning that the error should be bounded by the first neglected term. When terms up to second order are kept, the error is of order $\chi_3^{(+)} \approx .000125\chi_0^{(+)} \approx .00025$ in the example discussed here. There is a second expansion parameter involved in terms like ω_1 or ω_2, namely $1/ka$, where a is a length parameter that characterizes derivatives of the potential, i.e., $|\nabla V(r)| \approx 1/a|V(r)|$. Provided that the potential is sufficiently smooth, these corrections also are small. Thus, the expansion can produce accurate wave functions for electron scattering when the energy is sufficiently high and the potential is sufficiently smooth. One must be aware that the eikonal wave functions omit the wave bending effects that arise from exact solutions and are less accurate far from the nucleus. However, the eikonal wave functions can provide an accurate description of Coulomb distortion effects in the region where the nuclear charge density is nonzero.

It is possible to reduce the terms in the eikonal expansion to analytical forms for the potential

$$V(r) = -\frac{\alpha Z}{\sqrt{r^2 + R^2}}, \tag{48}$$

where $\alpha = e^2/(\hbar c)$, Z is the nuclear charge and R is a length parameter. This Coulombic potential corresponds to a charge density

$$\rho(r) = \frac{3Ze}{4\pi}\frac{R^2}{(r^2 + R^2)^{5/2}}. \tag{49}$$

The analytical results are given in Appendix A.

Figure 1 shows the eikonal phases for a charge $Z = 100$, electron energy $E = 200$ MeV and radius parameter $R = 2$ fermi. These parameters are chosen in order to make the corrections visible. The corrections are much smaller for a 500 MeV electron and smaller nuclear charge.

D. Focusing factors

As noted above, the focusing factors are important when Coulomb distorted waves are used. Ignoring for

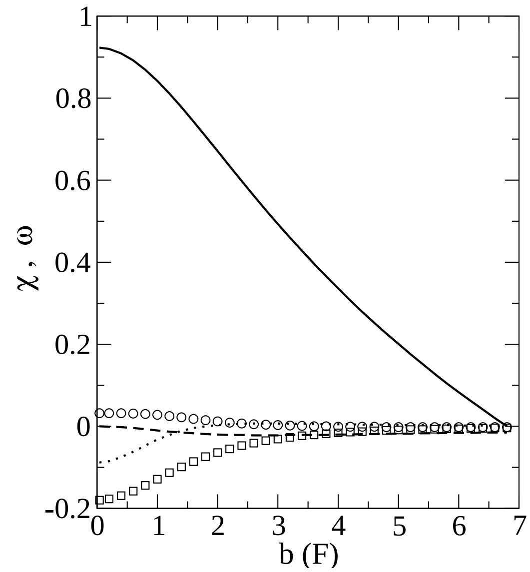

FIG. 1: Eikonal phases at z=0 versus impact parameter: solid line shows χ_0, dashed line shows χ_1, dotted line shows χ_2, rectangles show ω_1 and ovals show ω_2. A constant has been added to χ_0 such that it vanishes at $b = 3.5R$. Phases are shown for Z=100, E=200 MeV and R=2 fermi.

the moment the lower component spinors, we consider a current matrix element for emission of a photon of energy $\omega = E_i - E_f$ and momentum $\mathbf{q}$ using upper-component spinors corresponding to initial momentum $\mathbf{k}_i$, initial helicity λ_i, final momentum $\mathbf{k}_f$ and final helicity λ_f. The momentum transfer is $\mathbf{Q} = \mathbf{k}_i - \mathbf{k}_f$ and the current matrix element is

$$\begin{aligned} M^\mu &= \int d^3r u^{(-)*}_{\lambda_f}(\mathbf{r})\gamma^\mu e^{-i\mathbf{q}\cdot\mathbf{r}} u^{(+)}_{\lambda_i}(\mathbf{r}) \\ &= \int d^3r \xi^\dagger_{\lambda_f} e^{i(\mathbf{Q}-\mathbf{q})\cdot\mathbf{r}} e^{i\chi} e^{i\sigma_{e_f}\bar{\gamma}^{(-)*}_f} f^{D(-)}_f \gamma^\mu f^{D(+)}_i \times \\ & e^{i\sigma_{e_i}\bar{\gamma}^{(+)}_i} \xi_{\lambda_i}, \end{aligned} \tag{50}$$

where $\chi = \chi^{(-)}_f(\mathbf{r}) + \chi^{(+)}_i(\mathbf{r})$ and the Dirac focusing factors are defined by

$$\begin{aligned} f^{D(+)}_i(\mathbf{r}) &= \left(1 - \frac{V}{E_{2i}}\right)^{1/2} e^{-\omega^{(+)}_i}, \\ f^{D(-)}_f(\mathbf{r}) &= \left(1 - \frac{V}{E_{2f}}\right)^{1/2} e^{-\omega^{(-)}_f}. \end{aligned} \tag{51}$$

Note that $\chi^{(+)}_i$, $\omega^{(+)}_i$, and $\bar{\gamma}^{(+)}_i = \gamma^{(+)}_i + i\delta^{(+)}_i$ are obtained from Eqs. (36) to (37) with the z-axis parallel to initial momentum $\mathbf{k}_i$, while $\chi^{(-)}_f$, $\omega^{(-)}_i$, and $\bar{\gamma}^{(-)}_i = \gamma^{(-)}_i - i\delta^{(-)}_i$, are obtained from the same equations using the outgoing-wave condition ($\int_z^\infty$) with the z-direction parallel to final momentum $\mathbf{k}_f$.

Focusing factors differ from those appropriate to a Klein-Gordon wave function by the factor $(1 - V/E_{2i})^{1/2}$ as may be seen by comparing Eqs. (26) and (51). The $e^{-\omega^{(+)}_i} \approx 1 - V(0)/(2E_i)$ factor is the same to leading order for a Klein-Gordon and a Dirac wave function. Combining this part of the eikonal correction with the $(1 - V/E_{2i})^{1/2}$ yields a focusing factor $f^{D(+)}_i \approx 1 - V/E_i$ in the Dirac wave function, thus reproducing at $r = 0$ the expected factor $1 - V(0)/E_i$ that has been derived by Yennie, Boos and Ravenhall [23] based on a WKB analysis of the Dirac-Coulomb wave function. A similar result holds for the final-state focusing factor, $f^{D(-)}_f$, which is approximately $1 - V/E_f$. Thus, the overall focusing effect in the matrix element is approximately equal to $(1 - V(0)/E_f)(1 - V(0)/E_i)$, which is similar to the overall effect in the Klein-Gordan matrix element, Eq. (29). The reason that focusing factors did not appear in early analyses based on the eikonal approximation is because they arise from corrections to the eikonal approximation.

In passing, we note that the Glauber approximation is obtained when the eikonal phases for initial and final states are evaluated using for each a z-axis parallel to the average momentum, $\frac{1}{2}(\mathbf{k}_i + \mathbf{k}_f)$, and only the leading-order phases, $\chi^{(+)}_0$ and $\chi^{(-)}_0$, are retained. This approximation omits the focusing factors.

IV. ELECTRON CURRENT MATRIX ELEMENTS

Electron scattering involves the current matrix element

$$j^\mu_e = \int d^3r \bar{\Psi}^{(-)}_{\mathbf{k}_f\lambda_f}(\mathbf{r})\gamma^\mu e^{-i\mathbf{q}\cdot\mathbf{r}} \Psi^{(+)}_{\mathbf{k}_i\lambda_i}(\mathbf{r}), \tag{52}$$

where $\mathbf{q}$ is the three-momentum of a photon emitted at point $\mathbf{r}$, $\omega = E_i - E_f$ is the energy of the photon and

$$\Psi^{(+)}_{\mathbf{k}_i\lambda_i}(\mathbf{r}) = \frac{1}{\sqrt{2}} \begin{pmatrix} u^{(+)}_{\lambda_i}(\mathbf{r}) \\ \ell^{(+)}_{\lambda_i}(\mathbf{r}) \end{pmatrix}, \tag{53}$$

is a Dirac-Coulomb wave with a normalization factor $1/\sqrt{2}$ included in order that it reduces as $V \to 0$ to the $m = 0$ plane-wave spinor,

$$u_{\lambda_i}(\mathbf{k}_i) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \sigma \cdot \hat{k}_i \end{pmatrix} \xi_{\lambda_i} e^{i\mathbf{k}_i\cdot\mathbf{r}}. \tag{54}$$

Using lower-component spinors from Eq. (41) and the conventions of Bjorken and Drell [41] for the γ^μ matrices, one readily finds that the matrix element j^0_e involves the overall factor $1 + 4\lambda_f\lambda_i = 2\delta_{\lambda_f\lambda_i}$ and the matrix element $\mathbf{j}_e$ involves the overall factor $2\lambda_f + 2\lambda_i = (2\lambda_i)2\delta_{\lambda_f\lambda_i}$. Thus, helicity is conserved as it must be in the $m = 0$ limit. The electron current matrix elements are further reduced by use of the convention of Kubis [42] for the helicity matrix elements between initial and final states, leading to

$$j^\mu_e = \delta_{\lambda_f\lambda_i} \int d^3r e^{i(\mathbf{Q}-\mathbf{q})\cdot\mathbf{r}} e^{i\chi} f^{D(-)}_f(\mathbf{r}) f^{D(+)}_i(\mathbf{r}) h^\mu_e(\mathbf{r}) \tag{55}$$

where $h_e^\mu(\mathbf{r})$ is a four-vector of helicity matrix elements,

$$\begin{aligned}\{h_e^0, \mathbf{h}_e\} &= \xi^\dagger_{\lambda_f}(\theta_e) e^{i\sigma_{e_f}\bar{\gamma}_f^{(-)*}}\{1,\vec{\sigma}\} e^{i\sigma_{e_i}\bar{\gamma}_i^{(-)}}\xi_{\lambda_i} \\ &= \xi^\dagger_{\lambda_f}(\theta_e)\Big[cos\bar{\gamma}_f^{(-)*} + i\sigma_{e_f} sin\bar{\gamma}_f^{(-)*}\Big]\{1,\vec{\sigma}\} \\ &\quad \Big[cos\bar{\gamma}_i^{(+)} + i\sigma_{e_i} sin\bar{\gamma}_i^{(+)}\Big]\xi_{\lambda_i}. \end{aligned} \tag{56}$$

Here $\xi^\dagger_{\lambda_f}(\theta_e) = \xi^\dagger_{\lambda_f} e^{i\sigma_y\theta_e/2}$ is the helicity eigenstate for the outgoing electron and θ_e is the scattering angle of the electron. We find that

$$\begin{aligned} i\sigma_{e_i}\xi_{\lambda_i} &= 2\lambda_i e^{2i\lambda_i\phi_i}\xi_{-\lambda_i}, \\ \xi^\dagger_{\lambda_f}(\theta_e) i\sigma_{e_f} &= -2\lambda_f e^{-2i\lambda_f\phi_f}\xi^\dagger_{-\lambda_f}(\theta_e). \end{aligned} \tag{57}$$

Angle ϕ_i is the polar angle of initial state impact parameter, i.e., $\mathbf{b}_i = cos\phi_i\hat{x} + sin\phi_i\hat{y}$ and ϕ_f is the polar angle of the final state impact parameter, where initial and final momenta are in the xz-plane: $\mathbf{k}_i = k_i\hat{z}$ and $\mathbf{k}_f = k_f[cos\theta_e\hat{z} + sin\theta_e\hat{x}]$. Moreover the required helicity matrix elements are

$$\begin{aligned} \xi^\dagger_{\lambda_f}(\theta_e)\xi_{\lambda_i} &= \delta_{\lambda_f\lambda_i} cos\frac{1}{2}\theta_e + (\lambda_f - \lambda_i) sin\frac{1}{2}\theta_e \\ \xi^\dagger_{\lambda_f}(\theta_e)\;\vec{\sigma}\;\xi_{\lambda_i} &= (\lambda_f + \lambda_i)\Big(\hat{e}_{2\lambda_i} sin\frac{1}{2}\theta_e + \hat{e}_z cos\frac{1}{2}\theta_e\Big) \\ &\quad + |\lambda_f - \lambda_i|\Big(\hat{e}_{2\lambda_i} cos\frac{1}{2}\theta_e - \hat{e}_z sin\frac{1}{2}\theta_e\Big) \end{aligned} \tag{58}$$

where $\hat{e}_{2\lambda} = \hat{e}_x + (2\lambda)i\hat{e}_y$.

Combining these parts produces the required helicity matrix elements for the components of the current (plane-wave values are shown following the arrows),

$$\begin{aligned} h_e^0 &= A_{2\lambda_i} cos\frac{1}{2}\theta_e + C_{2\lambda_i} sin\frac{1}{2}\theta_e \longrightarrow cos\frac{1}{2}\theta_e, \\ h_e^x &= B_{2\lambda_i} sin\frac{1}{2}\theta_e + D_{2\lambda_i} cos\frac{1}{2}\theta_e \longrightarrow sin\frac{1}{2}\theta_e, \\ h_e^y &= 2i\lambda_i\Big(A_{2\lambda_i} sin\frac{1}{2}\theta_e - C_{2\lambda_i} cos\frac{1}{2}\theta_e\Big) \longrightarrow 2i\lambda_i sin\frac{1}{2}\theta_e, \\ h_e^z &= B_{2\lambda_i} cos\frac{1}{2}\theta_e - D_{2\lambda_i} sin\frac{1}{2}\theta_e \longrightarrow cos\frac{1}{2}\theta_e, \end{aligned} \tag{59}$$

where the spin-orbit parts of the eikonal phases enter the helicity matrix elements in the following four combinations

$$\begin{aligned} A_{2\lambda_i} &\equiv cos\bar{\gamma}_f^{(-)*} cos\bar{\gamma}_i^{(+)} - sin\bar{\gamma}_i^{(+)} sin\bar{\gamma}_f^{(-)*} e^{2i\lambda_i(\phi_i-\phi_f)}, \\ B_{2\lambda_i} &\equiv cos\bar{\gamma}_f^{(-)*} cos\bar{\gamma}_i^{(+)} + sin\bar{\gamma}_i^{(+)} sin\bar{\gamma}_f^{(-)*} e^{2i\lambda_i(\phi_i-\phi_f)}, \\ C_{2\lambda_i} &\equiv cos\bar{\gamma}_f^{(-)*} sin\bar{\gamma}_i^{(+)} e^{2i\lambda_i\phi_i} + cos\bar{\gamma}_i^{(+)} sin\bar{\gamma}_f^{(-)*} e^{-2i\lambda_i\phi_f}, \\ D_{2\lambda_i} &\equiv cos\bar{\gamma}_f^{(-)*} sin\bar{\gamma}_i^{(+)} e^{2i\lambda_i\phi_i} - cos\bar{\gamma}_i^{(+)} sin\bar{\gamma}_f^{(-)*} e^{-2i\lambda_i\phi_f}. \end{aligned} \tag{60}$$

Spin-orbit phases are of order $1/k$ so the corrections that they produce are less important at higher electron energy. The current matrix element given above is a general form expressed in terms of the complex phase $\overline{\gamma}^{(\pm)} = \gamma^{(\pm)} \mp i\delta^{(\pm)}$ and based on the zero-mass limit for the lower components of the Dirac wave function. One may work at various orders of approximation by using the results of Section III for the eikonal phases. The leading order contributions are from $\gamma_1^{(\pm)}$, which is real.

Some insight into the helicity matrix elements can be obtained by evaluating them for forward scattering and backward scattering. For forward scattering, the impact parameters and azimuthal angles for initial- and final-state eikonal phases are equal, i.e., $b_f = b_i$ and $\phi_f = \phi_i$. Evaluating the expression of Eq. (60) one finds the following simpler forms,

$$\begin{aligned} A_{2\lambda_i} &= cos(\bar{\gamma}_f^{(-)*} + \bar{\gamma}_i^{(+)}), \\ B_{2\lambda_i} &= cos(\bar{\gamma}_i^{(+)} - \bar{\gamma}_f^{(-)*}), \\ C_{2\lambda_i} &= \pm sin(\bar{\gamma}_f^{(-)*} + \bar{\gamma}_i^{(+)}), \\ D_{2\lambda_i} &= \pm sin(\bar{\gamma}_f^{(-)*} - \bar{\gamma}_i^{(+)}), \end{aligned} \tag{61}$$

where the upper sign applies for $\phi_i = \phi_f = 0$ and the lower one for $\phi_i = \phi_f = \pi$. The helicity matrix elements for the same values of ϕ_i are

$$\begin{aligned} h_e^0 &\to cos\Big(\frac{1}{2}\theta_e \mp (\bar{\gamma}_i^{(+)} + \bar{\gamma}_f^{(-)*})\Big), \\ h_e^x &\to sin\Big(\frac{1}{2}\theta_e \pm (\bar{\gamma}_i^{(+)} - \bar{\gamma}_f^{(-)*})\Big), \\ h_e^y &\to (2i\lambda_i) sin\Big(\frac{1}{2}\theta_e \mp (\bar{\gamma}_i^{(+)} + \bar{\gamma}_f^{(-)*})\Big), \\ h_e^z &\to cos\Big(\frac{1}{2}\theta_e \pm (\bar{\gamma}_i^{(+)} - \bar{\gamma}_f^{(-)*})\Big). \end{aligned} \tag{62}$$

Note that the helicity matrix elements are similar to the plane-wave matrix elements except for a shift of the electron scattering angle. The shift depends upon the azimuthal angle ϕ_i and as indicated by the $\pm$ signs, the shifts at $\phi_i = \pi$ are opposite to those at $\phi_i = 0$. Cancellations are expected in the integration over ϕ_i. These shifts that occur due to the spin-orbit interaction affect the longitudinal and transverse parts of the current in somewhat different ways. They may provide interesting insight into the accuracy with which one may make the L/T separation in the presence of Coulomb corrections. However, the required numerical evaluation is beyond the scope of this paper.

V. QUASI-ELASTIC ELECTRON SCATTERING BY NUCLEI

In this section we consider quasi-elastic scattering of electrons by nuclei but only taking into account the longitudinal current and the Coulomb corrections that arise from spin-independent terms in the eikonal expansion. The relevant matrix element involves one-photon exchange between the electron and a nucleon in the nucleus and the cross section takes a well-known form,

$$\frac{d\sigma}{d\Omega_f dE_f} = \int d\Omega_p \frac{4\alpha^2}{(2\pi)^5} \overline{|\mathcal{M}|^2} E_f^2 p E_p, \qquad (63)$$

where p is the momentum of the knocked-out nucleon and $E_p = \sqrt{M^2 + p^2}$ is its energy. The bar denotes an average over initial helicities and a sum over final helicities. The quasi-elastic matrix element is

$$\mathcal{M} = \delta_{\lambda_f \lambda_i} \int d^3r \int \frac{d^3q}{(2\pi)^2} e^{i(\mathbf{Q}-\mathbf{q})\cdot\mathbf{r}} e^{i\chi(\mathbf{r})} \times$$
$$f_f^{D(-)}(\mathbf{r}) f_i^{D(+)}(\mathbf{r}) h_e^{\mu}(\mathbf{r}) \left(\frac{1}{\mathbf{q}^2 - \omega^2} \right) J_{\mu}^{N}(\mathbf{q}, \mathbf{p}). \qquad (64)$$

In the plane-wave impulse approximation (PWIA), Coulomb distortion of the electron waves is neglected so the integration over $\mathbf{r}$ produces $\delta^{(3)}(\mathbf{q} - \mathbf{Q})$. The matrix element simplifies to

$$\mathcal{M}^{PWIA} = \delta_{\lambda_f \lambda_i} \frac{h^{\mu}_{PWIA} J_{N\mu}(\mathbf{Q}, \mathbf{p})}{Q^2}, \qquad (65)$$

where h^{μ}_{PWIA} denotes the helicity factors shown in Eq. (59) after the arrows. The PWIA cross section may be expressed in terms of longitudinal and transverse response functions, R_L and R_T, as follows

$$\frac{d\sigma}{d\Omega_f dE_f} = \sigma_{\text{Mott}} \left\{ \frac{Q^4}{\mathbf{Q}^4} R_L + \frac{Q^2}{2\mathbf{Q}^2} \frac{1}{\epsilon} \; R_T \right\}, \qquad (66)$$

where

$$\sigma_{\text{Mott}} = \frac{4\alpha^2 E_f^2 cos^2 \frac{\theta_e}{2}}{Q^4}, \qquad (67)$$

and

$$\epsilon = \left[1 + \frac{2\mathbf{Q}^2}{Q^2} tan^2 \frac{\theta_e}{2} \right]^{-1}. \qquad (68)$$

With Coulomb corrections included, the longitudinal matrix element of interest must take a gauge invariant form. This requires that the electron current must be conserved in the sense that

$$\int d^3r \Psi_{k_f}^{(-)*}(\mathbf{r}) \Big(\omega j_e^0 - \mathbf{q}\cdot\mathbf{j}_e \Big) e^{-i\mathbf{q}\cdot\mathbf{r}} \Psi_{k_i}^{(+)}(\mathbf{r}) = 0, \quad (69)$$

and that the nuclear current should separately be conserved,

$$\int d^3r \Psi_p^{(-)*}(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}} \Big(\omega J_N^0 - \mathbf{q}\cdot\mathbf{J}_N \Big) \psi(\mathbf{r}) = 0, \qquad (70)$$

where $\mathbf{q}$ is the photon three momentum. With Coulomb distorted waves, the photon momentum $\mathbf{q}$ differs from the electron's momentum transfer $\mathbf{Q} = \mathbf{k}_i - \mathbf{k}_f$ and the longitudinal current is defined with respect to the direction of the photon that is exchanged, not with respect to the difference of asymptotic electron momenta. Current conservation follows because the current obeys a Ward identity similar in form to Eq. (31).

Owing to current conservation, the longitudinal current matrix element can be simplified as follows,

$$j_e^0 J_N^0 - (\hat{q}\cdot\mathbf{j}^e)(\hat{q}\cdot\mathbf{J}_N) = j_e^0 J_N^0 \left(1 - \frac{\omega^2}{\mathbf{q}^2} \right), \qquad (71)$$

which holds either for the Klein-Gordon or Dirac case.

When the Dirac equation is used for the electron's Coulomb distorted waves, the matrix element due to the longitudinal current is

$$\mathcal{M}_L = \; = \; \delta_{\lambda_f \lambda_i} \int d^3r \int \frac{d^3q}{(2\pi)^2} e^{i(\mathbf{Q}-\mathbf{q})\cdot\mathbf{r}} e^{i\chi(\mathbf{r})} \times$$
$$f_f^{D(-)}(\mathbf{r}) f_i^{D(+)}(\mathbf{r}) h_e^0(\mathbf{r}) \left(\frac{1}{\mathbf{q}^2} \right) J_0^N(\mathbf{q}, \mathbf{p}) (72)$$

where Eq. (71) has been used to include the components of $\mathbf{j}_e$ and $\mathbf{J}_N$ that are parallel to $\mathbf{q}$. The longitudinal response function is obtained by dividing the cross section integrated over the angles of the knocked-out nucleon by the Mott cross section,

$$R_L = \frac{\mathbf{Q}^4}{\sigma_{Mott} Q^4} \int d\Omega_p \frac{4\alpha^2}{(2\pi)^5} \overline{|\mathcal{M}_L|^2} E_f^2 p E_p, \qquad (73)$$

where $\mathcal{M}_L$ is the longitudinal amplitude of Eq. (72). The full calculation thus involves a six-dimensional integration in order to obtain the amplitude $\mathcal{M}_L$. Two more integrations over the angles of the knocked-out nucleon are required in order to obtain the response function. Results based on the eight-dimensional integration are called "full calculations" in the following sections.

When the Klein-Gordon equation is used for the electron's Coulomb distorted waves, the time component of the current is $j^0 = [E_i + E_f - 2V(r)]/\sqrt{4E_i E_f}$ when Coulomb effects are included and $j^0 = (E_i + E_f)/\sqrt{4E_i E_f}$ in the PWIA. The matrix element in the Klein-Gordon case is

$$\mathcal{M}_L^{KG} = \int d^3r \int \frac{d^3q}{(2\pi)^2} e^{i(\mathbf{Q}-\mathbf{q})\cdot\mathbf{r}} e^{i\chi(\mathbf{r})} f_f^{KG(-)}(\mathbf{r}) \times$$
$$f_i^{KG(+)}(\mathbf{r}) \frac{[E_i + E_f - 2V(r)]}{\sqrt{4E_i E_f}} \left(\frac{1}{\mathbf{q}^2} \right) J_0^N(\mathbf{q}, \mathbf{p}), \qquad (74)$$

An analog of the Mott cross section based on the longitudinal current is

$$\sigma^{KG} = \frac{\alpha^2 E_f (E_i + E_f)^2}{E_i \mathbf{Q}^4} \qquad (75)$$

and the response function is

$$R_L = \frac{1}{\sigma^{KG}} \int d\Omega_p \frac{4\alpha^2}{(2\pi)^5} \overline{|\mathcal{M}_L^{KG}|^2} E_f^2 p E_p. \qquad (76)$$

The longitudinal respsonse function R_L is calculated using a very simple model of the nuclear current as follows,

$$J_N^\mu(\mathbf{q},\mathbf{p}) = \left(\frac{p_i^\mu + p_f^\mu}{\sqrt{4E_p(E_p-\omega)}}\right)\psi(\mathbf{q}-\mathbf{p}), \quad (77)$$

where $\psi(\mathbf{k})$ is a gaussian wave function for a bound nucleon,

$$\hat{\psi}(\mathbf{k}) = (2\pi\beta^2)^{3/4} e^{-\beta^2 k^2/4}, \quad (78)$$

normalized such that $\int d^3k |\psi(\mathbf{k})|^2/(2\pi)^3 = 1$. This simple model is used because the Coulomb corrections should depend mainly on the electron wave functions. Calculations are based on the value $\beta = 2$ fermi.

The nuclear current given above is based upon initial and final momenta,

$$p_f^\mu = \big(E_p,\ \mathbf{p}\big), \qquad p_i^\mu = \big(E_p-\omega,\ \mathbf{p}-\mathbf{q}\big), \quad (79)$$

where ω and $\mathbf{q}$ are the photon's energy and momentum. Because of energy conservation, $E_p = M + \omega - B$, where $B \approx .008GeV$ is a typical binding energy of a nucleon. Gauge-invariance must hold so Eq. (70) is used to eliminate the component of the nuclear current that is parallel to the photon's momentum.

In the plane-wave impulse approximation, the longitudinal response function is

$$R_L^{PWIA} = \frac{pE_p}{(2\pi)^5}\int d\Omega_p \left|\frac{\mathbf{Q}^2\mathcal{M}_L}{cos\frac{\theta_e}{2}}\right|^2 \quad (80)$$

where the longitudinal amplitude is

$$\mathcal{M}_L^{PWIA} = \delta_{\lambda_f\lambda_i}\frac{h^0_{PWIA}J_N^0(\mathbf{Q},\mathbf{p})}{\mathbf{Q}^2}. \quad (81)$$

Using the current and wave function described above leads to

$$R_L^{PWIA} = \frac{p}{(2\pi)^3}\frac{(2E_p-\omega)^2}{4(E_p-\omega)}\int d\Omega_p |\hat{\psi}(\mathbf{Q}-\mathbf{p})|^2. \quad (82)$$

The angular integrations are straightforward, yielding

$$R_L^{PWIA}(\omega,\mathbf{Q}) = \frac{1}{\sqrt{2\pi}}\frac{(2E_p-\omega)^2}{4(E_p-\omega)}\frac{\beta}{|\mathbf{Q}|}\times \left(e^{-\beta^2(|\mathbf{Q}|-p)^2/2} - e^{-\beta^2(|\mathbf{Q}|+p)^2/2}\right). \quad (83)$$

Here the $PWIA$ response function is normalized so that at fixed $\mathbf{Q}$, $\int d\omega R_L(\mathbf{Q},\omega) \approx 1$.

VI. EFFECTIVE MOMENTUM APPROXIMATION REVISITED

As shown by Rosenfelder [25] and Traini [26], there are significant cancellations in the Coulomb corrections when response functions are evaluated in an effective-momentum approximation (*ema*). This approximation usually is based on expanding the eikonal phase in a Taylor's series about $\mathbf{r} = \mathbf{0}$ and keeping the first two terms as follows,

$$\chi(\mathbf{r}) \approx \chi(\mathbf{0}) + \mathbf{r}\cdot\nabla\chi(\mathbf{0} + \cdots . \quad (84)$$

The focusing factors are approximated by their values at $\mathbf{r} = \mathbf{0}$ and the helicity matrix elements are approximated by the plane-wave values. Integration over $\mathbf{r}$ then gives $\delta^{(3)}(\mathbf{q}-\mathbf{Q}_{eff})$, so the longitudinal amplitude simplifies to the PWIA form as follows,

$$\mathcal{M}_L^{ema} = 2\pi\delta_{\lambda_f\lambda_i}h^0_{PWIA}J_N^0(\mathbf{Q}_{eff},\mathbf{p})\frac{f_f^{D(-)}(\mathbf{0})f_i^{D(+)}(\mathbf{0})}{\mathbf{Q}^2_{eff}}. \quad (85)$$

The effective momentum involves the gradient of the eikonal phase shift $\chi = \chi_f^{(-)} + \chi_i^{(+)}$ at the origin. Because of cylindrical symmetry of $\chi_i^{(+)}$ about the direction $\hat{k}_i$, $\nabla\chi_i^{(+)}$ at the origin is nonzero only along the direction $\hat{k}_i$, and similarly $\nabla\chi_f^{(-)}$ at the origin is nonzero only along the direction $\hat{k}_f$. With $v_i = v_f \approx 1$, we find the same result as Traini,

$$\mathbf{Q}_{eff} = \hat{k}_i\Big[k_i - \delta k\Big] - \hat{k}_f\Big[k_f - \delta k\Big], \quad (86)$$

where $\delta k = V(0)$. It is correct up to first order in the eikonal expansion because the contribution from the gradient of eikonal correction χ_1 vanishes at the origin.

The analyses of Rosenfelder [25] and Traini [26] are based on the approximate focusing factors, $f_i^{D(+)} \approx 1 - V(0)/E_i$ and $f_f^{D(-)} \approx 1 - V(0)/E_f$. Coulomb effects in the focusing factors and the effective photon propagator cancel if one considers the photon propagator of the transverse amplitude, which is $1/[\mathbf{Q}^2_{eff} - \omega^2] = 1/[4[k_i - V(0)][k_f - V(0)]sin^2\frac{1}{2}\theta_e]$, i.e.,

$$\frac{f_f^{D(-)}(\mathbf{0})f_i^{D(+)}(\mathbf{0})}{\mathbf{Q}^2_{eff} - \omega^2} = \frac{1}{Q^2}, \quad (87)$$

which is the same as in the plane-wave case, Eq. (65). However, the requirements of gauge invariance that have been incorporated into the longitudinal matrix element of Eq. (72) show that the effective photon propagator is $1/\mathbf{Q}^2_{eff}$. In that case the cancellations still are significant but not perfect. We find

$$\frac{f_f^{D(-)}(\mathbf{0})f_i^{D(+)}(\mathbf{0})}{\mathbf{Q}^2_{eff}} = \frac{1}{\mathbf{Q}^2 + \Delta}, \quad (88)$$

where Δ represents a Coulomb correction to the plane-wave result,

$$\Delta = \omega^2 V(0) \frac{[k_i + k_f - V(0)]}{[k_i - V(0)][k_f - V(0)]} \qquad (89)$$

As an example, for 500 MeV electrons scattering from a 25 MeV Coulomb potential with θ_e=60^o, $\Delta/\mathbf{Q}^2 \approx .01$ at $\omega = .125$, which is close to the quasi-elastic peak.

Numerical calculations based on partial-wave expansions of Dirac-Coulomb waves also indicate that the Coulomb corrections do not cancel to the extent that Eq. (87) would suggest.[43] For the e^+ and $e-$ response functions the *ema* analysis suggests that the same response function should be obtained if the energy is shifted such that $E_i(e^+) + V(0) = E_i(e^-) - V(0)$, where $V(0)$ is the electron-nucleus potential at $r = 0$. This shift results in the same $\mathbf{Q}_{eff}$ for $e+$ and $e-$ scattering. Using the gauge-invariant response function, the correction Δ has opposite sign for e^+ scattering than for e^- scattering. It can produce a 4% difference in the e^+ and e^- longitudinal response functions, whereas there would be no difference based on Eq. (87). Numerical calculations based on partial-wave expansions of Dirac-Coulomb waves indicate that the Coulomb corrections do not cancel to the extent that Eq. (87) would suggest. In their analysis based on the full DWBA, Kim *et al.* [44] have obtained results for the sum of longitudinal and transverse responses, which differ by about 15% for 420 MeV $e+$ and 383 MeV $e-$.

VII. NUMERICAL CALCULATIONS FOR QUASI-ELASTIC SCATTERING

Calculations of the longitudinal response function are performed for three cases: PWIA, ema and the full calculation. We also use distorted waves based on the Dirac equation and the Klein-Gordon equation. The calculations are based on a charge $Z = 38$ and radius $R = 2$ fermi in the Coulomb potential of Eq. (48). Eikonal phases are evaluated through second order, i.e., $\chi = \chi_0 + \chi_1 + \chi_2$ and $\omega = \omega_1 + \omega_2$. However, the expansion converges rapidly for the parameters and energies used and results based on $\chi_0 + \chi_1$ and ω_1 differ by about 0.3% at the quasi-elastic peak.

In order to check the accuracy of the full calculation of R_L, which involves an eight-dimensional integration, we have performed calculations with $Z = 0$ and compared with the analytical $PWIA$ result. Results are accurate to between 0.5% to 2% with the integration points that have been used.

Figure 2 and Table I show the longitudinal response function for 500 MeV electrons with scattering angle $\theta_e = 60^o$ based on the use of the Dirac current and distorted waves. In Fig. 2 the full calculation is shown by circles, the *ema* calculation based on $\delta k = V(0)$ is shown by the solid line and the PWIA calculation is shown by the dotted line. Figure 3 shows the response function for the same kinematics using the Klein-Gordon current and distorted waves. The results are quite close to those based on the Dirac equation, which is expected because of the similarity of eikonal wave functions for the two cases when the spin-dependent phases are omitted. As noted earlier, the Klein-Gordon and Dirac results involve similar overall focusing factors when consistent currents and wave functions are used. We also show by the dashed line in Fig. 3 a calculation following the prescription

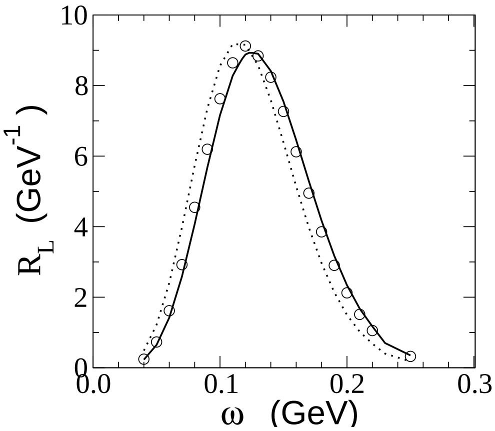

FIG. 2: Longitudinal response function versus the electron's energy loss, ω, calculated using Dirac distorted waves for e^- scattering at $E = 500MeV$ and $\theta_e = 60^o$. Dotted line shows $PWIA$, solid line shows *ema* and circles show full calculations based on Eq. (72).

TABLE I: Numerical results for R_L for 500 MeV electrons scattered by 60^o.

ω GeV	PWIA	*ema*	Full
0.040	0.49	0.23	0.24
0.050	1.22	0.65	0.73
0.060	2.41	1.42	1.62
0.070	3.98	2.59	2.92
0.080	5.73	4.07	4.55
0.090	7.35	5.68	6.20
0.100	8.56	7.16	7.63
0.110	9.18	8.28	8.65
0.120	9.16	8.87	9.12
0.130	8.57	8.90	8.84
0.140	7.59	8.42	8.24
0.150	6.39	7.55	7.27
0.160	5.14	6.45	6.12
0.170	3.98	5.29	4.95
0.180	2.96	4.17	3.85
0.190	2.14	3.17	2.90
0.200	1.50	2.34	2.12
0.210	1.02	1.68	1.51
0.220	0.68	1.17	1.06

of Ref [35] in which the Klein-Gordon current is used with the Dirac focusing factors. This produces a significantly different result because of the extraneous factor $1 - V(0)/\bar{E}$ that is included. The reason why the extra factor produces a large change is because the other focusing factors largely cancel out of the matrix element as in Eqs. (87) or (88).

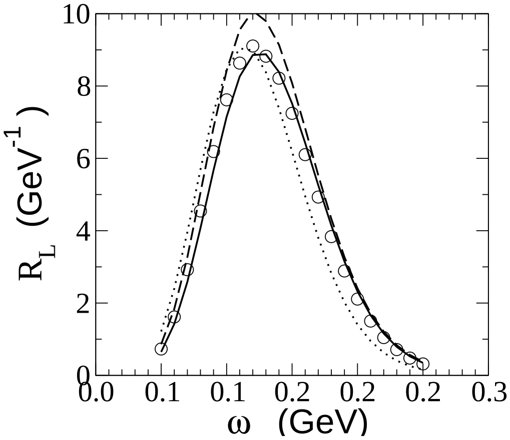

FIG. 3: Longitudinal response function versus the electron's energy loss, ω, calculated using Klein-Gordon distorted waves for e^- scattering at $E = 500MeV$ and $\theta_e = 60^o$. Dotted line shows $PWIA$, solid line shows ema, dashed line shows the prescription of Ref. [35] and circles show full calculations based on Eq. (74).

For the remainder of this section we use only the Dirac current and Dirac wave functions. Figure 4 shows the longitudinal response function for e^+ scattering at 540 MeV. In general the ema is seen in Figures 2 and 4 to produce a significant shift of R_L away from the PWIA result and towards the full calculation of R_L. There also is good agreement between the response functions for $e-$ and $e+$ scattering at the energies that make $\mathbf{Q}_{eff}$ close to the same for both.

Upon closer inspection, we find that the ema result is not precise. Figures 5 and 6 show the ratios of the ema response functions to the full ones for e^- and e^+ scattering, respectively. Deviations of 5-10% occur at values of ω that are away from the quasi-elastic peak. In the literature one finds a variety of suggestions for improving the ema, such as using $\delta k = V(R)$ rather than $\delta k = V(0)$, where R is the mean nuclear radius, or using $\delta k = \overline{V}$, where $\overline{V}$ is the average potential within a sphere of radius R. These prescriptions improve the ema at some values of ω but not all values.

A precise form of the effective-momentum approximation would be useful for removing Coulomb corrections from experimental data in a straightforward manner. It would allow a determination of the PWIA response function. In order to have a precise result, one should determine appropriate values of the momentum-shift function $\delta k(k_i, \omega, \theta_e)$ from which the appropriate $\mathbf{Q}_{eff}$ may be calculated as in Eq. (86). In order to determine this

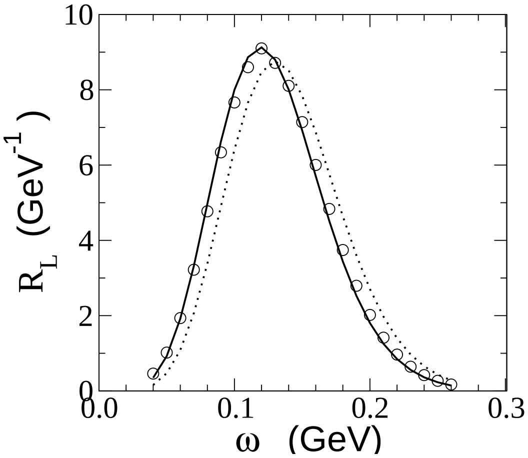

FIG. 4: Longitudinal response function versus the positron's energy loss, ω, for e^+ scattering at $E = 540MeV$ and $\theta_e = 60^o$. Dotted line shows $PWIA$, solid line shows ema and circles show full calculations based on Eq. (72).

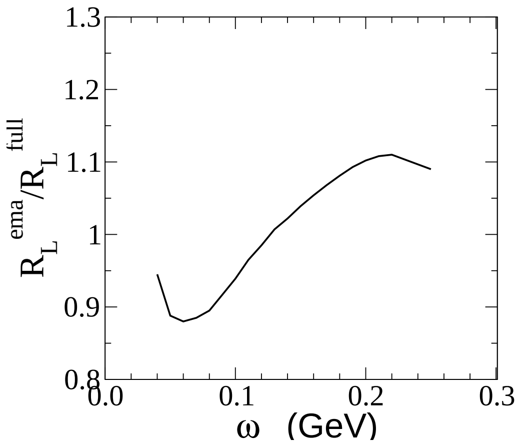

FIG. 5: Ratio R_L^{ema}/R_L as a function of energy loss ω for 500 MeV e^- scattering at angle $\theta_{e^-} = 60^o$.

function, we have fit the full response function as follows,

$$R_L(\mathbf{Q}, \omega) = A \; R_L^{PWIA}(\mathbf{Q}_{eff}, \omega) \tag{90}$$

at fixed values of k_i and θ_e by varying δk and the normalization constant A. The role of the normalization constant A is to ensure that $\delta k(\omega)$ is a smooth function. Without this parameter, δk can take anomalous values near the quasi-elastic peak. Figures 7 and 8 show the values of δk that are found to yield Eq. (90) for scattering of e^- (with $A_{e^-} = 0.987$) and e^+ (with $A_{e^+} = 0.973$). The sign of δk is in accord with the potential $V(0)$, which is -27 MeV for electrons and +27 MeV for positrons. However, a constant value of δk does not suffice. The required values vary with ω at a fixed value of $\theta_e = 60^o$, within a range $0.7|V(0)| < |\delta k| < 1.5|V(0)|$. Note that the shift δk required for the e^+ response function is not simply obtained by reversing the sign of the shift required for

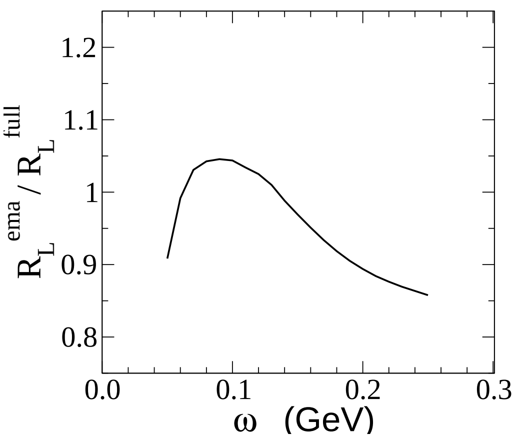

FIG. 6: Ratio R_L^{ema}/R_L as a function of energy loss ω for 540 MeV e^+ scattering at angle $\theta_e = 60^o$.

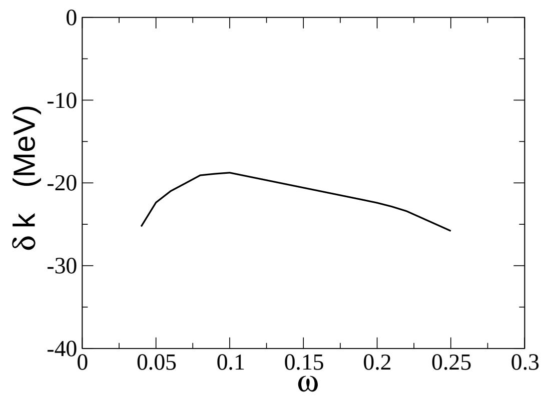

FIG. 7: Shift δk used in $\mathbf{Q}_{eff}$ in order to achieve the equality of Eq. (90), as a function of electron's energy loss, ω, for 500 MeV e^- scattering at $\theta_{e^-} = 60^o$. The normalization constant is $A = 0.987$.

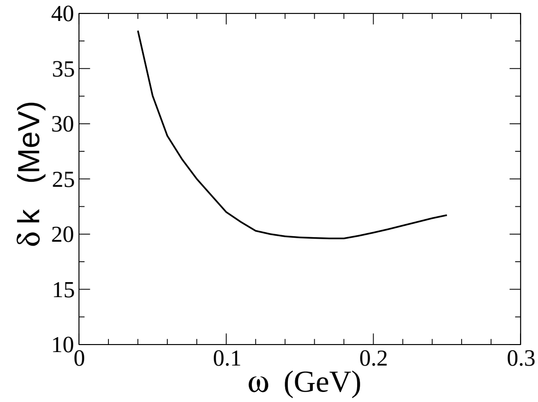

FIG. 8: Shift δk used in $\mathbf{Q}_{eff}$ in order to achieve the equality of Eq. (90), as a function of positron's energy loss, ω, for 540 MeV e^+ scattering at $\theta_{e^+} = 60^o$. The normalization constant is $A = 0.973$.

the e^- response function. Nevertheless, the closeness of A to unity and δk to $V(0)$ suggests that an *ema* analysis using the fit values of δk would be well-motivated on physical grounds.

If the momentum-shift function and normalization constant A are determined theoretically for a given nucleus based on a sophisticated model of the nuclear current, and R_L values are available based upon experimental data, our results suggest that one may use Eq. (90) with the experimentally determined R_L in order to extract information about the undistorted response function, R_L^{PWIA}. Of course, the accuracy would depend upon the accuracy of the L/T separation.

VIII. SUMMARY

In this paper, we develop the eikonal expansion for relativistic wave functions based on the Klein-Gordon equation and the Dirac equation, each with a Coulombic potential. The purpose is to obtain some insight into the Coulomb corrections in quasi-elastic electron scattering without a significant loss of accuracy. The eikonal expansion is carried out to obtain corrections up to order $1/k^2$ to the eikonal approximation. We show that focusing factors are obtained in a systematic manner by use of the eikonal expansion. Although focusing factors take somewhat different forms for the Klein-Gordon and Dirac wave functions, equivalent results are obtained for the current matrix elements for the two cases because of the Coulomb correction to the Klein-Gordon current.

Based on a simple form of the Coulomb potential, analytical results are given for the eikonal phases and thus for the eikonal wave functions. For scattering of electrons with energies of a few hundred MeV or more, the eikonal expansion converges rapidly.

Coulomb corrections in quasi-elastic electron scattering are considered using the analytical wave functions based on the eikonal expansion. For the longitudinal response function, we show that the approximate evaluation of the matrix element using the effective-momentum approximation is modified by the requirements of gauge invariance. That modification causes a small but significant difference between the e^- and e^+ response functions. Moreover, the *ema* is not sufficiently accurate to allow a precise analysis of data because the effective momenta $k_i - \delta k$ and $k_f - \delta k$ are not precise when δk is taken to be independent of ω, as it is in the usual form of *ema*. Using a simple model of the nuclear current, we find that use of a function $\delta k(\omega)$ can yield a precise form of the *ema*. The analysis should be repeated for more sophisticated models of the nuclear current with the goal of determining the function $\delta k(\omega)$ for different models. Based on our results, we suggest that if one is able to extract $R_L(\omega, E, \theta_e)$ from experimental data at fixed electron beam energy, E, and fixed electron scattering angle, θ_e, then it may be equated to a constant $A \approx 1$ times the PWIA response function evaluated at an effective momentum transfer.

Whether one may extract the longitudinal response with reasonable accuracy has not been resolved in this work. We have addressed the transverse current matrix elements and have shown how the spin-dependent Coulomb corrections modify the plane-wave helicity matrix elements. There is a potentially interesting effect in the manner that Coulomb corrections enter as a shift of the electron scattering angle in helicity matrix elements. However, these spin-dependent Coulomb corrections have not been included in our calculations. Evaluating their effects may help to provide insight into the accuracy with which longitudinal and transverse response functions can be separated. Because we have only calculated the response functions based on the longitudinal-current matrix elements, our results cannot be compared with experimental data, or with a full DWBA analysis, because both include an inseparable sum of longitudinal and transverse responses. In particular, we note that the effects of Coulomb corrections in the helicity matrix elements of Eq. (62) would go in different directions for $e+$ and $e-$ scattering.

Acknowledgments

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APPENDIX A: ANALYTICAL PHASES

Analytical results are given for the eikonal phase shifts based on the potential of Eq. (48). Terms of order m^2/E^2 are omitted. In order to give the analytical results in a reasonably short form, we define the following quantities: $\eta = Z\alpha/v$, $u = \sqrt{r^2+R^2}$, $w = \sqrt{b^2+R^2}$ and $\beta = tan^{-1}(w/z)$, where β varies from 0 to π as z varies from $-\infty$ to ∞. In $\chi_0^{(+)}$, a term $\eta ln(2\Lambda)$ is omitted, where the eikonal integral is calculated over the range $\{-\Lambda, z\}$. This contribution provides an irrelevant phase factor that occurs for potentials that behave as $1/r$ as $r \rightarrow \infty$.

The analytical results are

$$\chi_0^{(+)} = \eta ln\left(\frac{z+u}{w^2}\right) \tag{A1}$$

$$\chi_1^{(+)} = -\frac{\eta^2 b^2}{kw^4}\left(z+u+\frac{1}{2}w\beta\right) \tag{A2}$$

$$\chi_2^{(+)} = -\frac{\eta^3 b^2(z+u)}{k^2 w^6}\Bigg[\frac{R^2}{2u} + \frac{(R^2-b^2)(z+u)}{w^2} + \frac{(2R^2-b^2)\beta}{2w}\Bigg] + \frac{\eta R^2}{2k^2 w^4}\Big(\frac{z+u}{u} + \frac{w^2 z}{2u^3}\Big)$$
$$-\frac{2\eta R^2}{k^2 w^6}\Bigg[\Big(1-\frac{3b^2}{w^2}\Big)\Big(z(z+u) + \frac{w^2}{4u}(u-z)\Big) + \frac{b^2(z+u)}{4u} + \frac{b^2 w^2 z}{8u^3}\Bigg] + \frac{1}{4EE_2}\nabla^2\chi_0^{(+)}, \quad \text{(A3)}$$

$$\omega_1^{(+)} = -\frac{\eta R^2}{k w^4}\left[z+u-\frac{w^2}{2u}\right] \quad \text{(A4)}$$

$$\omega_2^{(+)} = -\frac{\eta^2}{k^2 w^4}\Bigg\{\frac{b^2}{4}\left(1+\frac{z}{u}\right)^2 + \left(1-\frac{6b^2R^2}{w^4}\right)\times$$
$$\left[z(z+u) + w^2 ln[2u(z+u)/w^2] + zw\beta - \frac{w^2}{2}\right]$$
$$+b^2\Big(1-\frac{2b^2}{w^2}\Big)\times\left[2ln[2u(z+u)/w^2] - 1 + \frac{z\beta}{w}\right]$$
$$+b^2\Bigg[ln[2u(z+u)/w^2] + \frac{b^2}{2u(z+u)} + \frac{z\beta}{2w}\left(1-\frac{b^2}{2w^2}\right) - \frac{1}{2} - \frac{3b^2}{4w^2} + \frac{b^2}{4u^2}\Bigg]\Bigg\}$$
$$+\frac{\eta^2 R^2}{k^2 w^4}\Bigg\{\Big(1-\frac{2b^2}{w^2}\Big)ln[2u(u+z)/w^2] - \frac{w^2}{4u^2} + \frac{4b^2}{w^4}\Big[z(u+z) + \frac{w^2}{2}\Big] - \frac{b^2(u+z)}{2uw^2} + \frac{b^2}{4u^2}\Bigg\} \quad \text{(A5)}$$

$$\gamma_1^{(+)} = -\frac{\eta b}{2E_2 w^2}\left(\frac{z+u}{u}\right) \quad \text{(A6)}$$

$$\gamma_2^{(+)} = -\frac{\eta^2 b}{2kE_2 w^4}\left[2\Big(1-\frac{2b^2}{w^2}\Big)\Big(z+u+\frac{w\beta}{2}\Big) + \frac{b^2}{u} + \frac{b^2\beta}{2w} + \frac{b^2 z}{2u^2}\right] \quad \text{(A7)}$$

$$\delta_2^{(+)} = \frac{2\eta R^2 b}{kE_2 w^6}\Big[z+u-\frac{w^2}{2u}-\frac{w^4}{8u^3}\Big] \quad \text{(A8)}$$

The quantity $\nabla^2\chi_0^{(+)}$ that occurs in $\chi_2^{(+)}$ for the Dirac case is given by

$$\nabla^2\chi_0^{(+)} = \frac{\eta R^2}{u^3}\left[\frac{z+2u}{(z+u)^2} - \frac{4u^3}{w^4}\right] \quad \text{(A9)}$$

That term should be omitted for the Klein-Gordon case.

Corresponding results for the incoming wave boundary condition are obtained from Eq. (24), which holds for any of the quantities in Eqs. (A1) to (A9).

APPENDIX B: EIKONAL EXPANSION FOR DIRAC WAVE FUNCTION

For $z \to -\infty$, the incoming wave should reduce to a plane wave, which is realized by the boundary conditions $\bar{\chi}^{(+)}, \bar{\gamma}^{(+)} \to 0$ as $z \to -\infty$. Inserting Eq. (35) into Eq. (34) leads to

$$\Bigg(E_1 - V - \frac{(k\hat{z}+\nabla\bar{\chi}^{(+)})^2}{E_2-V} - \sigma\cdot\mathbf{p}\frac{1}{E_2-V}\sigma\cdot(k\hat{z}+\nabla\bar{\chi}^{(+)}) - \sigma\cdot(k\hat{z}+\nabla\bar{\chi}^{(+)})\frac{1}{E_2-V}\sigma\cdot\mathbf{p} - \sigma\cdot\mathbf{p}\frac{1}{E_2-V}\sigma\cdot\mathbf{p}\Bigg)(1-V/E_2)^{1/2}e^{i\sigma_e\bar{\gamma}^{(+)}} = 0.\quad\text{(B1)}$$

Momentum operators in this expression act on all quantities to their right. Performing the indicated differentiations of the factor $(1-V/E_2)^{1/2}$ and multiplying by $E_2^{1/2}(E_2-V)^{1/2}/(2k)$ leads to

$$\Bigg(-\frac{\partial\bar{\chi}^{(+)}}{\partial z} - \frac{V_c}{v} - \frac{(\nabla\bar{\chi}^{(+)})^2}{2k} + i\frac{\nabla^2\bar{\chi}^{(+)}}{2k} - \frac{\sigma_e V_s^{(+)}}{2k} + i\frac{\partial}{\partial z} + i\frac{\nabla\bar{\chi}^{(+)}}{k}\cdot\nabla - \frac{\nabla V}{2k(E_2-V)}\cdot\nabla + \frac{\sigma\cdot\nabla V}{2k(E_2-V)}\sigma\cdot\nabla + \frac{\nabla^2}{2k}\Bigg)e^{i\sigma_e\bar{\gamma}^{(+)}} = 0, \quad \text{(B2)}$$

where $v = k/E \approx 1$ and we have defined central and spin-orbit potentials as follows,

$$V_c(r) = V(r) - \frac{V^2(r)}{2E} + \frac{\nabla^2 V(r)}{4E(E_2-V(r))} + \frac{3(\nabla V(r))^2}{8E(E_2-V(r))^2}, \quad \text{(B3)}$$

and

$$V_s^{(+)}(b,z) = \frac{\partial V(r)}{\partial b} - \frac{1}{E_2-V}\frac{\partial V(r)}{\partial z}\frac{\partial\bar{\chi}^{(+)}(\mathbf{r})}{\partial b}. \quad \text{(B4)}$$

The spin-orbit term in Eq. (B2) involving potential $V_s^{(+)}$ gives rise to the spin-dependent eikonal phase $\bar{\gamma}^{(+)}$.

Some care is required in evaluating the derivatives of the spin-dependent phase factor, for example, $\nabla\sigma_e = -(\sigma_b/b)\hat{e}$ and $\nabla^2\sigma_e = -\sigma_e/b^2$. We find

$$-\frac{\partial\bar{\chi}^{(+)}}{\partial z} - \frac{V_c}{v} - \frac{(\nabla\bar{\chi}^{(+)})^2}{2k} + i\frac{\nabla^2\bar{\chi}^{(+)}}{2k} - \frac{\sigma_e V_s^{(+)}}{2k} - \frac{\partial\bar{\gamma}^{(+)}}{\partial z}\sigma_e - \frac{\nabla\bar{\chi}^{(+)}\cdot\nabla\bar{\gamma}^{(+)}}{k}\sigma_e - i\frac{\nabla V\cdot\nabla\bar{\gamma}^{(+)}}{2k(E_2-V)}\sigma_e$$
$$+X_3 + i\frac{\nabla^2\bar{\gamma}^{(+)}}{2k}\sigma_e - \frac{(\nabla\bar{\gamma}^{(+)})^2}{2k} - \frac{i}{2kb^2}sin(\bar{\gamma}^{(+)})\sigma_e e^{-i\sigma_e\bar{\gamma}^{(+)}} = 0, \quad \text{(B5)}$$

where an overall factor $e^{i\sigma_e\bar{\gamma}^{(+)}}$ is omitted, and

$$X_3 = e^{-i\bar{\gamma}^{(+)}\sigma_e}\frac{\sigma\cdot\nabla V}{2k(E_2-V)}\sigma\cdot\nabla e^{i\bar{\gamma}^{(+)}\sigma_e} \quad (B6)$$

The quantity X_3 is found to be of order $1/k^3$ or smaller so will be dropped in the following.

The part of Eq. (B5) that involves σ_e and the part that does not involve σ_e must vanish separately. Using trace techniques to project out these parts produces equations for the phases $\bar{\chi}^{(+)}$ and $\bar{\gamma}^{(+)}$ as follows,

$$\begin{aligned}\frac{\partial\bar{\chi}^{(+)}}{\partial z} &= -\frac{V_c}{v} - \frac{(\nabla\bar{\chi}^{(+)})^2}{2k} + i\frac{\nabla^2\bar{\chi}^{(+)}}{2k} - \frac{(\nabla\bar{\gamma}^{(+)})^2}{2k} \\ &- \frac{1}{2kb^2}\sin^2(\bar{\gamma}^{(+)}),\end{aligned} \quad (B7)$$

$$\begin{aligned}\frac{\partial\bar{\gamma}^{(+)}}{\partial z} &= -\frac{V_s^{(+)}}{2k} - \frac{\nabla\bar{\chi}^{(+)}\cdot\nabla\bar{\gamma}^{(+)}}{k} + i\frac{\nabla^2\bar{\gamma}^{(+)}}{2k} \\ &- \frac{i}{4kb^2}\sin(2\bar{\gamma}^{(+)}) - i\frac{\nabla V\cdot\nabla\bar{\gamma}^{(+)}}{2k(E_2-V)}.\end{aligned} \quad (B8)$$

One may see from the second of these equations that $\bar{\gamma}^{(+)}$ is of order $1/k$ or smaller. Thus, the last two terms in Eq. (B7), the last term in Eq. (B8) and X_3 are of order $1/k^3$ or smaller. We omit these terms in the following with the objective of obtaining results correct to order $1/k^2$. We also simplify $\sin(2\bar{\gamma}^{(+)}) \approx 2\bar{\gamma}^{(+)}$ for the same reason.

Integration in accord with the outgoing-wave boundary conditions produces the basic equations

$$\begin{aligned}\bar{\chi}^{(+)}(\mathbf{r}) &= -\int_{-\infty}^{z} dz'\frac{V_c(\mathbf{r}')}{v} - \int_{-\infty}^{z} dz'\frac{(\nabla'\bar{\chi}^{(+)}(\mathbf{r}'))^2}{2k} \\ &+i\int_{-\infty}^{z} dz'\frac{\nabla'^2\bar{\chi}^{(+)}(\mathbf{r}')}{2k},\end{aligned} \quad (B9)$$

$$\begin{aligned}\bar{\gamma}^{(+)}(\mathbf{r}) &= -\int_{-\infty}^{z} dz'\frac{V_s^{(+)}(\mathbf{r}')}{2k} \\ &- \int_{-\infty}^{z} dz'\frac{\nabla'\bar{\chi}^{(+)}(\mathbf{r}')\cdot\nabla'\bar{\gamma}^{(+)}(\mathbf{r}')}{2k} \\ &+ i\int_{-\infty}^{z} dz'\frac{\left[\nabla'^2 - 1/b^2\right]\bar{\gamma}^{(+)}(\mathbf{r}')}{2k},\end{aligned} \quad (B10)$$

where $\mathbf{r}' = (\mathbf{b}, z')$.

With incoming-wave boundary conditions, the wave function must become a plane wave as $z \to +\infty$ and is written as

$$u^{(-)}(\mathbf{r}) = \left(1-\frac{V}{E_2}\right)^{1/2} e^{ikz}e^{-i\bar{\chi}^{(-)}}e^{-i\sigma_e\bar{\gamma}^{(-)}}. \quad (B11)$$

and the complex phases $\bar{\chi}^{(-)}(\mathbf{r})$ and $\bar{\gamma}^{(-)}(\mathbf{r})$ obey

$$\begin{aligned}\bar{\chi}^{(-)}(\mathbf{r}) &= -\int_{z}^{\infty} dz'\frac{V_c(\mathbf{r}')}{v} - \int_{z}^{\infty} dz'\frac{(\nabla'\bar{\chi}^{(-)}(\mathbf{r}'))^2}{2k} \\ &- i\int_{z}^{\infty} dz'\frac{\nabla'^2\bar{\chi}^{(-)}(\mathbf{r}')}{2k}\end{aligned} \quad (B12)$$

$$\begin{aligned}\bar{\gamma}^{(-)}(\mathbf{r}) &= -\int_{z}^{\infty} dz'\frac{V_s^{(-)}(\mathbf{r}')}{2k} \\ &- \int_{z}^{\infty} dz'\frac{\nabla'\bar{\chi}^{(-)}(\mathbf{r}')\cdot\nabla'\bar{\gamma}^{(-)}(\mathbf{r}')}{2k} \\ &-i\int_{z}^{\infty} dz'\frac{\left[\nabla'^2 - 1/b^2\right]\bar{\gamma}^{(-)}(\mathbf{r}')}{2k}.\end{aligned} \quad (B13)$$

where $V_s^{(-)}$ is

$$V_s^{(-)}(b,z) = \frac{\partial V(r)}{\partial b} + \frac{1}{E_2-V}\frac{\partial V(r)}{\partial z}\frac{\partial\bar{\chi}^{(-)}(\mathbf{r})}{\partial b}. \quad (B14)$$

The eikonal expansion is the iterative solution appropriate to large k of Eqs. (B9) and (B10) or Eqs. (B12) and (B13). The expansion for $\bar{\chi}^{(+)}$ initially takes the form

$$\bar{\chi}^{(+)} = \bar{\chi}_0^{(+)} + \bar{\chi}_1^{(+)} + \bar{\chi}_2^{(+)} + \cdots, \quad (B15)$$

where each term in the series is smaller than the previous one. The "barred" phases are in general complex with the exception that $\bar{\chi}_0^{(+)}$ is real when the potential is real.

Lowest order terms may be obtained from Eqs. (B9) and (B12) by keeping just the leading terms on the right side,

$$\begin{aligned}\bar{\chi}_0^{(+)}(\mathbf{r}) &= -\frac{1}{v}\int_{-\infty}^{z} dz' V_c(r'), \\ \bar{\chi}_0^{(-)}(\mathbf{r}) &= -\frac{1}{v}\int_{z}^{\infty} dz' V_c(r').\end{aligned} \quad (B16)$$

It should be noted that the same symbol $\bar{\chi}_0^{(+)}$ was used in the Klein-Gordon case but the meaning is different here because of the extra terms in $V_c(r)$, Eq. (B3). Using the same iterative procedure that was used to obtain Eqs. (10) and (11) leads to

$$\begin{aligned}\bar{\chi}_1^{(+)}(\mathbf{r}) &= -\frac{1}{2k}\int_{-\infty}^{z} dz'(\nabla'\bar{\chi}_0^{(+)}(\mathbf{r}'))^2 \\ &+ \frac{i}{2k}\int_{-\infty}^{z} dz'\nabla'^2\bar{\chi}_0^{(+)}(\mathbf{r}'),\end{aligned} \quad (B17)$$

$$\begin{aligned}\bar{\chi}_2^{(+)}(\mathbf{r}) &= -\frac{1}{2k}\int_{-\infty}^{z} dz'\Big[2\nabla'\bar{\chi}_0^{(+)}(\mathbf{r}')\cdot\nabla'\bar{\chi}_1^{(+)}(\mathbf{r}') \\ &+ (\nabla'\bar{\chi}_1^{(+)}(\mathbf{r}'))^2\Big] + \frac{i}{2k}\int_{-\infty}^{z} dz'\nabla'^2\bar{\chi}_1^{(+)}(\mathbf{r}').\end{aligned} \quad (B18)$$

For the spin-dependent phase, $\bar{\gamma}^{(+)}$, the iterative expansion is

$$\bar{\gamma}^{(+)} = \bar{\gamma}_1^{(+)} + \bar{\gamma}_2^{(+)} + \cdots \tag{B19}$$

where

$$\bar{\gamma}_1^{(+)}(\mathbf{r}) = -\frac{1}{2k}\int_{-\infty}^{z} dz' V_s^{(+)}(\mathbf{b}, z'), \tag{B20}$$

$$\begin{aligned} \bar{\gamma}_2^{(+)}(\mathbf{r}) \; &= \; -\frac{1}{2k}\int_{-\infty}^{z} dz' \nabla' \bar{\chi}^{(+)}(\mathbf{r}') \cdot \nabla' \bar{\gamma}_1^{(+)}(\mathbf{r}') \\ &+ \; \frac{i}{2k}\int_{-\infty}^{z} dz' \Big(\nabla'^2 - \frac{1}{b^2}\Big) \bar{\gamma}_1^{(+)}(\mathbf{r}'). \end{aligned} \tag{B21}$$

Complex eikonal phases $\bar{\chi}^{(+)}$ and $\bar{\gamma}^{(+)}$ are decomposed into real and imaginary parts (the "unbarred" phases) as follows

$$\begin{aligned} \bar{\chi}^{(+)} &= \chi^{(+)} + i\omega^{(+)} \\ \bar{\gamma}^{(+)} &= \gamma^{(+)} + i\delta^{(+)}. \end{aligned} \tag{B22}$$

Because the central and spin-dependent potentials have terms that involve various powers of $1/k$, the "barred" phases that are obtained from the iterative expansion above are not ordered systematically. Therefore a second expansion is performed to obtain terms that are proportional to powers of $1/k$. This leads to Eqs. (36) to (37) of the text.